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The Rate of Divergence of Hypercubic Pyramidal Elements

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Abstract. The earliest papers on applications of the finite element method for solving partial differential equations were related to one-dimensional problems. In the present day, the finite element method has been successfully applied for solving boundary and eigenvalue problems in higher-dimensional Euclidean spaces. The minimal measure of degeneracy of the simplicial elements depends on the dimension of the finite space. It tends to infinity when the dimension of the Euclidean space grows up unbounded. The pyramidal elements are relatively new compared to hexahedral and simplicial elements. The major role of the pyramids is to assure conforming coupling between structured and unstructured finite element meshes. This paper deals with the rate of divergence of the sequence of pyramidal finite elements. Detailed proofs of two divergence theorems are obtained. For illustrations, the results of the theorems are presented graphically.

Keywords: rate of divergence, measure of degeneracy, n-dimensional pyramidal elements, canonical domains.

Mathematics Subject Classification: 52B11, 65N30

1. Introduction

The pioneering paper on three-dimensional pyramidal elements has been written by Bedrosian [2] way back in the early nineties. The role of the pyramidal finite elements as transitional elements between hexahedral and simplicial meshes has been clarified by Ainsworth and Fu in [1]. The pyramidal elements have been an object of great interest in the last decades [3,8,9,10,12]. The computational cost of each kind of element determines the location of the corresponding elements in the composite finite element triangulation. The hypercubic elements are located in the interior subdomain, the k-pyramids are in the interface subdomain and the simplicial elements are used in the boundary layer. Basic results on the n-dimensional finite element method have been published by Brandts et al. [5,7]. In [6], the authors have given the answer to the question of why the finite element method can be used in multidimensional Euclidean spaces. Properties of n-dimensional simplicial finite elements have been studied by Brandts et al. [6] and Petrov and Todorov [11]. The rate of divergence generated by various sequences of simplicial elements has been determined by Petrov and Todorov in the latter paper.

This paper is devoted to the rate of divergence of sequences of pyramidal elements. Due to considerable practical importance, we study the case of canonical pyramidal elements in a separate theorem. Additionally, we consider the increment of the degeneracy measure depending on the height of the elements. Rigorous proofs of the divergence theorems are the major contributions of the paper. Dependence of the degeneracy measure on the height of the elements and the dimension of the Euclidean spaces is presented graphically. The interface subdomains are always canonical or at least parallelotopial even in the case when the original domain is curved and with a complex geometry. Therefore, irregular pyramidal elements are beyond our consideration.

2. Divergence analysis of the pyramidal elements

The space $\mathbf{R}^n$ is the n-dimensional Euclidean space provided with the norm $||\cdot||$. Let P be a convex polytope in $\mathbf{R}^n$.We denote: the diameter of P by $d(P)$; the n-dimensional volume of P by $\mathrm{vol}_n(P)$; the set of all vertices of P by $V(P)$; and the center of gravity of the polytope by $G(P)$.

Definition 1 *The degeneracy measure of an arbitrary convex polytope* P *is equal to*

$$\delta(P) = \frac{\mathrm{vol}_{n-1}(\partial P) d(P)}{2n\mathrm{vol}_n(P)}. \tag{1}$$

This measure has been used in a finite element analysis of simplicial elements by Bey [4].

Definition 2 Petrov and Todorov [11]. *Let* $\{T^n\}$, *be a sequence of polytopes in* $\mathbf{R}^\infty$. *The number* $\alpha(\{T^n\})$ *defined by* $\delta(T^n) = O(n^\alpha)$ *is called the rate of divergence for the sequence* $\{T^n\}$.

Each n-dimensional hypercube can be tessellated by $2n$ regular $(n-1)$-hypercubic pyramids. For instance, each tesseract can be tessellated by eight regular cubic pyramids.

Definition 3 *The regular* $(n-1)$*-hypercubic pyramids that tessellate an n-dimensional hypercube are called canonical*.

Let C^n be an n-dimensional unit hypercube

$$\hat{C}^{\mathrm{n}} = \{\hat{x} \in \mathbf{R}^n \mid 0 \leq x_i \leq 1, \;\; i = 1,2,\dots,n\}, \;\; n \in \mathbf{N}.$$

The denotation

$$P^n = [p_{1,} p_{1,} \dots, p_m]$$

stands for an n-dimensional polytope with vertices $p_i, i = 1,2,\dots,m$. We denote the class of all polytopes that are geometrically similar to the polytope P by $[P]$.

The pyramid

$$\hat{P}^n = \left[V\left(\hat{C}^{n-1}\right), \left(\hat{C}^n\right)\right]$$

is chosen for the reference element.

Concerning a hypercubic pyramid P, we introduce the following denotations:

$A(P)$ is the apex of P;
$B(P)$ is the base of P;
$H(P)$ is the length of the height of P;
$h(P)$ is the length of the slant height of P;
$L(P)$ is the base length;
$l(P)$ is the length of the lateral edge of P.

The canonical pyramids play an important role in the multidimensional finite element method. That is why, we study the divergence rate α_P of the sequence $\{P^n\}, P^n \in [\hat{P}^n]$ by the next theorem.

Theorem 1 *The degeneracy measure* $\delta(P^n)$, $P^n \in \left[\hat{P}^n\right]$, generated by the canonical pyramidal elements is

$$\delta(P^n) = \sqrt{n-1}\left(1+\sqrt{2}\right).$$

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Proof. In the first part of the proof, we obtain that

$$\mathrm{vol}_{n-1}(\partial\hat{P}^n) = 1+\sqrt{2},\ \ \forall n \geq 2. \tag{2}$$

All lateral facets of the n-dimensional pyramidal element $\hat{P}^n$ are identical, therefore they have the same volume. To calculate $\mathrm{vol}_{n-1}(\partial\hat{P}^n)$ we consider a lateral facet

$$\hat{P}^{n-1} = \left[V\left(\hat{C}^{n-2}\right), G(\hat{C}^n)\right]$$

The height $h(\hat{P}^n)$ is independent of n due to

$$H(P^{n-1}) = \mathrm{dist}\left(G\left(\hat{C}^{n-2}\right), G(\hat{C}^n)\right)$$

$$= \mathrm{dist}\left(\left(\frac{1}{2},\frac{1}{2},\dots,\frac{1}{2},0,0\right),\left(\frac{1}{2},\frac{1}{2},\dots,\frac{1}{2}\right)\right) = \frac{\sqrt{2}}{2}.$$

The pyramid $\hat{P}^n$ has $2(n-1)$ lateral facets. Since $\mathrm{vol}_{n-1}\left(B\left(\hat{C}^{n-2}\right)\right) = 1$, therefore

$$\mathrm{vol}_{n-1}(P^{n-1}) = \frac{\sqrt{2}}{2(n-1)}.$$

and (2) is proved.

From the definition of $\hat{P}^n$ it follows that

$$\mathrm{vol}_n\left(\hat{P}^n\right) = \frac{1}{2n}\mathrm{vol}_n\left(\hat{C}^n\right) = \frac{1}{2n}. \tag{3}$$

On the other hand, $d\left(\hat{P}^n\right)$ is equal to the space diagonal of $\hat{C}^{n-1}$, i.e.

$$d\left(\hat{P}^n\right) = \sqrt{n-1}. \tag{4}$$

It remains to substitute (2), (3) and (4) in (1) in order to complete the proof.□

Theorem 1 guarantees that $\alpha_P = \frac{1}{2}$.

Remark 1 *From the proof of Theorem 1 it follows that* $\mathrm{vol}_{n-1}(\partial\hat{P}^n)$ *is independent of n.*

In the next theorem, we extend the result declared by Theorem 1. Theorem 2 describes the increment of the degeneracy measure by varying the height H of the regular pyramid P^n.

Theorem 2 *The sequence of regular hypercubic pyramids* $\{P^n(H)\}$ *has a rate of divergence*

$$\alpha(P^n(H)) = \frac{1}{2}, \forall H \geq 2.$$

Proof. Since all regular hypercubic pyramids with the same ratio $\frac{H(P)}{L(P)}$ has the same measure of degeneracy we can fix the base length to one and vary the height in order to study the rate of divergence of the sequence $\{P_n\}$. We choose a representative

$$\hat{P}^n(H) = \left[V(\hat{C}^{n-1}), A\left(\frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2}, H\right)\right]$$

of the class of the n-dimensional regular hypercubic pyramids. We consider a lateral facet

$$P^{n-1}(H) = \left[V(\hat{C}^{n-2}), A\right]$$

of the element $\hat{P}^n(H)$. The volume of the boundary

$$\mathrm{vol}_{n-1}\left(\partial \hat{P}^n(H)\right) = 2(n-1)\mathrm{vol}_{n-1}(P^{n-1}(H)) + \mathrm{vol}_{n-1}(B(P^{n-1}(H)))$$

$$= 2h\left(\hat{P}^n\right) + 1 = 2\mathrm{dist}\left(G\left(\hat{C}^{n-2}\right), A\left(\hat{P}^n(H)\right)\right) + 1$$

$$= 2\left\|\left(0,0, \dots 0,0, \frac{1}{2}, H\right)\right\| + 1 = \sqrt{1+4H^2} + 1.$$

The volume of the pyramidal element

$$\mathrm{vol}_n\left(\hat{P}^n(H)\right) = \frac{1}{n}\mathrm{vol}_{n-1}\left(B(\hat{P}^n(H))\right)H = \frac{H}{n}.$$

The next step is to calculate the diameter

$$d\left(\hat{P}^n(H)\right) = \max\left\{d\left(B\left(\hat{P}^n(H)\right)\right), l\left(\hat{P}^n(H)\right)\right\}$$

$$= \max\left\{\sqrt{n-1}, \left\|\left(-\frac{1}{2}, -\frac{1}{2}, \dots, -\frac{1}{2}, \frac{1}{2}, H\right)\right\|\right\}$$

$$= \max\left\{\sqrt{n-1}, \frac{1}{2}\sqrt{n-1+4H^2}\right\}.$$

Thus, for the measure of degeneracy, we obtain

$$\delta\left(\hat{P}^n(H)\right) = \begin{cases} \varphi(H)\sqrt{n-1}, & if\ H \le \frac{1}{2}\sqrt{3(n-1)} \\ \frac{1}{2}\varphi(H)\sqrt{n-1+4H^2}, & \text{otherwise} \end{cases},$$

where $\varphi(x) = \frac{1+\sqrt{1+4x^2}}{2x}$. The latter confirms the statement of the theorem. □

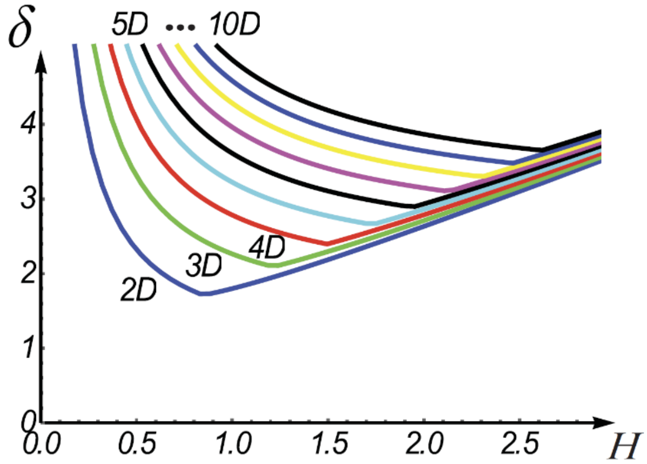

Figure 1. Dependence of the measure of degeneracy on H and n.

The results of Theorem 2 are presented graphically in Figure 1.

Let $\{R^n\}$ be the sequence of regular simplices in $\mathbf{R}^\infty$ with a rate of divergence α_{RS}. Then

$$\alpha_P < \alpha_{RS} \tag{5}$$

and

$$\delta(\hat{P}^n) = \delta(R^n), \quad \forall n \geq 10.$$

We emphasize the fact that

$$\delta(R^n) = \delta(\hat{P}^n), \quad \forall n < 10$$

despite (5).

3. Conclusion

The divergence rate of the sequence of pyramidal finite elements is determined. All kinds of regular hypercubic pyramidal elements are analyzed. A comparison between the divergence rate of the sequence of canonical pyramids and the one generated by the sequence of regular simplices is done. We establish that the rate of divergence in the regular pyramidal elements is twice lower than the optimal divergence rate for the simplicial elements. Despite this, the regular simplices have a better measure of degeneracy than the canonical pyramids for all n so that $2 \leq n \leq 9$. The $(n-1)$-dimensional volume of the boundary of each canonical hyperpyramid obtained from the n-dimensional unit hypercube is a constant. The phenomenon (2) is available in each Euclidean space with $n \geq 2$.

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