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Finite Element Method for the One-Dimensional Telegraph Equation

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Abstract. A linear telegraph equation with periodic boundary conditions is chosen for a model problem in this paper. The model problem is used in order to be illustrated a new finite element method for solving hyperbolic boundary value problems. There are various weak forms of the telegraph equation that have been in use up to now. An original weak problem related to the problem of interest is obtained. A detailed multigrid algorithm describing the essence of the considered method is developed. Numerical simulations including smoothing of an error function and a graph of an approximate solution are demonstrated. An approximate asymptotic rate of convergence is calculated by applying three successive triangulations and cubic trial functions.

1. Introduction and setting of the problem

There are too many papers [16, 19, 25, 27, 31, 35] that consider the initial-boundary value problem for telegraph equations in unbounded domains. But just a few researchers [13, 35] have studied a telegraph equation in bounded domains. The linear telegraph equation with doubly periodic boundary conditions has been investigated for well-posedness by Ortega and Robles-Pérez [26]. They have proved that the telegraph operator

$$Lu = u_{tt} - a u_{xx} + \beta u_t - \alpha(z) u, \qquad z = (x, t)$$

satisfies the maximum principle if the variable coefficient α is estimated by a function depending on the constant β and $a = 1$. Ortega and Robles-Pérez suppose that the coefficient α is a doubly periodic function. Additionally, the authors declare a unique solution of the linear telegraph equation when the maximum principle holds. On the other hand, they have clearly shown a class of coefficients in the telegraph operator that generate multiple solutions of the hyperbolic boundary value problem with periodic boundary conditions. Further, Mawhin [21] and Mawhin et al. [22, 23] have extended the results of Ortega and Robles-Pérez in various multidimensional cases. The periodic solutions of the telegraph equation have also been studied by Grossinho and Nkashama [12], and Kim [15]. They have independently investigated similar nonlinear telegraph equations. De Araújo et al. consider multidimensional telegraph equation with time-periodic and Dirichlet boundary conditions. They established a unique weak solution in the multidimensional nonlinear case. A nonlinear telegraph equation with more complicated time-periodic boundary conditions has been investigated by Kharibegashvili and Dzhokhadze in [14].

Various numerical methods have been applied for solving the telegraph equation in the last decades. The semidiscrete finite element method for the initial-boundary value problem has been analyzed by Larson and Thomée [16] in order to solve the wave equation. Another semidiscrete approach has been realized for the wave-diffusion equation by Chen et al. [8]. Such kind of methods can be successfully applied to solve the telegraph equation. The time-marching method is a kind of meshless numerical method used by Lin et al. [19] for solving the initial-boundary value problem. The wavelet solutions based on Haar basis functions have been analyzed in [5, 24]. B-Splines are the basic tools for solving the initial-boundary value problem for the second-order one-dimensional telegraph equation in the papers written by T. Nazir et al. [25]. The boundary element method [10], the finite difference method [2], and collocation methods [13] have also been applied for solving the telegraph equation. A space-time continuous Galerkin method has been investigated for convergence by Zhao et al. [35]. Chen et al. [8] have created a two-grid method for an initial hyperbolic boundary value problem in a bounded domain. The same finite element approach can be applied to the telegraph equation. We also have to mention some papers [6, 27, 31, 34], which present numerical simulations of the solutions of linear telegraph equations.

Most of the authors [25, 27, 35] solving the telegraph equation use separate triangulations in time and space.

We introduce some basic definitions and denotations. Let Ω be a rectangular domain

$$\Omega = \{z(x,t) \mid 0 < x < l, 0 < t < T\}.$$

The edges

$$\Gamma_0 = \{(x,0) \mid 0 < x < l\}$$

and

$$\Gamma_T = \{(x,T) \mid 0 < x < l\}$$

are used in the further analyses. The real Sobolev space $H^n(\Omega)$ for n nonnegative integer is provided with the norm $||\cdot||_{n,\Omega}$ and the seminorm $|\cdot|_{n,\Omega}$.

A completely different approach for solving the telegraph equation is created in this paper. The proposed method is an alternative to the finite difference methods and wavelet methods for hyperbolic boundary value problems. This paper is devoted to a finite element method for the one-dimensional telegraph equation

$$Lu = f \text{ in } \Omega,\ \alpha, f \in L^2(\Omega) \tag{1}$$

with the following boundary conditions

$$u(0,t) = u(l,t) = 0,\ t \in [0,T], \tag{2}$$

$$u(x,0) = u(x,T),\ x \in [0,l], \tag{3}$$

$$u_t(x,0) = u_t(x,T),\ x \in [0,l]. \tag{4}$$

Additionally, we suppose that a is a positive constant, β is a constant and

$$\alpha(z) \geq 0\ \forall z \in \Omega. \tag{5}$$

An original finite element method for solving hyperbolic boundary value problems is obtained. A one-dimensional linear telegraph equation with periodic boundary conditions is chosen for a model problem. Full space-time discretizations are used for solving the problem of interest. Different iterative methods are considered for solving the system of the finite element equations arising from the discretizations. A multigrid algorithm in the case of a singular system of finite element equations is

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described. Numerical simulations of the weak solution are presented. The major contribution of this paper is a completely new weak formulation of the original problem. This approach for obtaining the weak problem is not restricted only to the telegraph equation. It can be applied to other hyperbolic problems. The proposed theory is supported by numerical simulations. The approximate asymptotic rate of convergence (ARC) is calculated.

Further, the paper is organized as follows. An original weak form of the linear telegraph equation with periodic boundary conditions is obtained in Section 2. A finite element method for solving the strong problem is described in Section 3. Subsection 3.1 is devoted to finite element discretizations. Iterative solution methods are included in Subsection 3.2. A multigrid algorithm for solving the system of the finite element equations is described in pseudocode in the same subsection. Numerical simulations are discussed in Section 4. The paper ends with some concluding remarks.

2. A weak form of the problem of interest

Let

$$\hat{\Omega} = \{z(x,t) \mid 0 < x < \pi, 0 < t < 2\pi\}.$$

Kim [15] has considered the nonlinear telegraph equation

$$u_{tt} - u_{xx} + \beta u_t + \operatorname{sign}(u)|u|^p = f(z) \text{ in } \hat{\Omega},\ \beta \neq 0 \text{ and } p > 0 \tag{6}$$

with superlinear growth. The equation (6) is provided with the following boundary conditions

$$u(0,t) = u(\pi,t) = 0,\ t \in [0,2\pi], \tag{7}$$

$$u(x,0) = u(x,2\pi),\ x \in [0,\pi], \tag{8}$$

$$u_t(x,0) = u_t(x,2\pi),\ x \in [0,\pi]. \tag{9}$$

Kim has proved in [15] that the problem (6) with the boundary conditions (7-9) has a weak solution for all $f \in L^2(\Omega)$. The latter means that the problem (1-5) is well-posed having in mind that it can be obtained from (6-9) by $p = 1$ and $\Omega = \mathcal{F}(\hat{\Omega})$, where $\mathcal{F}$ is an affine transformation.

The application of the finite element method for solving a boundary value problem requires a weak formulation of the original problem. For this purpose, we define the space

$$\mathbf{V} = \{v \in H^1(\Omega) \mid v \text{ satisfies the boundary conditions (2), (3) and (4)}\}.$$

There are various weak formulations of the telegraph equation, see for instance Chen [8], Kim [15], Zhao et al. [35], etc. Analyzing the error in the weak solution Chen et al. [8] have essentially applied that

$$(v_{tt}, v_t) = \frac{1}{2}\frac{d}{dt}(v_t, v_t),$$

where $(\cdot,\cdot)$ is the $L^2(\Omega)$-scalar product. Zhao et al. [35] decrease the order of the equation by the substitution $u = v_t$. Thus they present the weak formulation by a system of equations depending on both variables u and v.

Here we use another way to construct the weak formulation. By multiplying both sides of (1) with $v \in \mathbf{V}$ and integrating by parts we obtain:

$$(u_{tt}, v) - (a u_{xx}, v) + (\beta\, u_t, v) + (\alpha\, u, v) = (f, v),$$

$$\big((1+a)u_{tt},v\big)-(a(u_{xx}+u_{tt}),v)+(\beta u_t,v)+(\alpha u,v)=(f,v),$$

$$\big((1+a)u_{tt},v\big)-(a\Delta u,v)+(\beta u_t,v)+(\alpha u,v)=(f,v),$$

$$-\big((1+a)u_t,v_t\big)+(a\nabla u,\nabla v)+(\beta u_t,v)+(\alpha u,v)=(f,v),$$

$$(a\nabla u,\nabla v)+(\alpha u,v)+2\left(\frac{\beta}{2}u_t,v\right)=\big((1+a)u_t,v_t\big)+(f,v),$$

$$(a\nabla u,\nabla v)+(\alpha u,v)=\big((1+a)u_t,v_t\big)+\frac{\beta}{2}\big((u,v_t)-(u_t,v)\big)+(f,v).$$

We define the following bilinear forms:

$$a(u,v)=\int_\Omega(a\nabla u\cdot\nabla v+\alpha uv)\,dz,\qquad dz=dxdt,$$

$$b(u,v)=(1+a)\int_\Omega u_tv_t\ dz,$$

$$c(u,v)=\int_\Omega\frac{\beta}{2}(uv_t-u_tv)dz.$$

The bilinear form $a(u,v)$ is elliptic and symmetric. The bilinear forms $b(u,v)$ and $c(u,v)$ are symmetric and skew-symmetric correspondingly. The ellipticity of the bilinear form $b(u,v)$ is not guaranteed. The problem

$$W:\begin{cases}\text{Find } u\in\mathbf{V} \text{ such that}\\ a(u,v)=b(u,v)+c(u,v)+(f,v)\quad\forall v\in\mathbf{V}.\end{cases}$$

is a weak formulation of the boundary value problem (1-5).

Remark 1 *The weak problem (W) can be successfully obtained with variable coefficients $a(z)$ and $\beta(z)$ if $a,\beta\in\mathbf{V}$.*

3. A finite element solution method

3.1 *Finite element discretizations*

Let τ_k $k=0,1,2,\ldots$ be a sequence of successive hierarchical finite element triangulations of the space-time domain Ω and $\mathbf{V}_k\subset\mathbf{V}$ be the corresponding finite element space. The space $\mathbf{V}_k$ is spanned by the nodal basis functions $\varphi_i^k(z)$, $i=1,2,\ldots,n_k=\dim\mathbf{V}_k$. The set $\mathcal{N}_k$ is composed of all nodes of τ_k belonging to $\Omega\cup\Gamma_0\cup\Gamma_T$. For the sake of simplicity, we restrict ourself to the simplicial Lagrangian finite elements and we suppose that the initial triangulation τ_0 is a uniform partition of the domain Ω.

A discrete problem (W_k) associated with the weak formulation (W) looks as follows

$$W_k:\begin{cases}\text{Find } u_k\in\mathbf{V}_k \text{ so that}\\ a(u_k,v_k)=b(u_k,v_k)+c(u_k,v_k)+(f,v_k)\ \ \forall v_k\in\mathbf{V}_k.\end{cases}$$

We search for a finite element solution

$$u_k(z) = U_k \cdot \Phi_k(z) = \sum_{i=1}^{n_k} U_{k,i}\varphi_i^k(z) \in \mathbf{V}_k,$$

where $U_{k,i} = u_k(a_i)$ and $\Phi_\mathrm{k}(z)$ is the vector of nodal basis functions.
Let

$$I^k : \mathbf{V} \to \mathbf{V}_k$$

be the Lagrangian nodal interpolation operator and

$$I_k : \mathbf{V}_{k-1} \to \mathbf{V}_k$$

defined by $I_k : v_{k-1} \to I^k v_{k-1}$ be the intergrid transfer operator from the finite element space $\mathbf{V}_{k-1}$ to $\mathbf{V}_k$. The operator P_k maps the finite element space $\mathbf{V}_k$ onto $\mathbf{R}^{n_k}$ by $P_k : v_k \to V_k$. We convert the discrete problem (W_k) in a matrix form

$$\widehat{W}_k : \begin{cases} \text{Find a } n_k - \text{dimensional vector } U_k \text{ so that} \\ \qquad A_k U_k = B_k U_k + C_k U_k + F_k. \end{cases}$$

in order to implement the method. The matrices in ($\widehat{W}_k$) are defined by:
$A_k = a\hat{A}_k + \widehat{M}_k$;
$\hat{A}_k$ is the stiffness matrix;
$\widehat{M}_k = \left\{\int_\Omega \alpha(z)\varphi_i^k\varphi_j^k\, dz,\ 1 \le i,j \le n_k\right\},\ dz = dxdt;$
$B_k = (1+a)\hat{B}_k;$
$C_k = \frac{\beta}{2}\hat{C}_k;$
$\hat{B}_k = \left\{\int_\Omega \frac{\partial\varphi_i^k}{\partial t}\frac{\partial\varphi_j^k}{\partial t} dz,\ 1 \le i,j \le n_k\right\};$
$\hat{C}_k = \left\{\int_\Omega \left(\varphi_i^k \frac{\partial\varphi_j^k}{\partial t} - \varphi_j^k \frac{\partial\varphi_i^k}{\partial t}\right) dz,\ 1 \le i,j \le n_k\right\};$
$F_k = \left\{\int_\Omega f\varphi_i^k\, dz,\ 1 \le i \le n_k\right\}.$
The matrix A_k is symmetric and positive definite due to (5). The matrices B_k and C_k are symmetric and skew-symmetric, which is essential from computational point of view.

3.2 *Iterative solution methods*
This section deals with the linear system

$$D_k U_k = F_k,\ D_k = A_k - (B_k + C_k) \tag{10}$$

arising from the problem (W_k). First, we consider the case of positive definite matrix D_k. Following Ghoussoub and Moradifam [11], we improve the system (10) in two stages.

First, we multiply both sides of (10) by D_k^T and obtain

$$\hat{Q}_k U_k = \hat{E}_k, \tag{11}$$

Where $\hat{Q}_k = D_k^T D_k$ and $\hat{E}_k = D_k^T F_k$.

The matrix $\widehat{D}_k = A_k - B_k$ is a symmetric part of D_k. If $\widehat{D}_k$ is invertible, we continue with the second stage. Multiplying both sides of (11) by the preconditioner $D_k^T \widehat{D}_k^{-1}$, we obtain

$$Q_k U_k = E_k, \tag{12}$$

where

$$Q_k = D_k^T \widehat{D}_k^{-1} \widehat{Q}_k \text{ and } E_k = D_k^T \widehat{D}_k^{-1} \widehat{E}_k.$$

Thus, we continue solving (12) instead of (10). The Fréchet derivatives

$$DJ_k(U_k)V_k = U_k Q_k V_k - F_k V_k$$

and

$$D^2 J_k(U_k)(V_k, V_k) = V_k Q_k V_k$$

of the functional

$$J_k(U_k) = \frac{1}{2} U_k Q_k U_k - F_k U_k$$

guarantee that J_k is convex with a unique stationary point U_k. To start a multigrid iteration, we solve the problem (W_0) in the coarsest triangulation τ_0 by any method. The approximate solution in the grid τ_k is denoted by $\hat{u}_k$. Let $\langle \cdot,\cdot \rangle$ be the inner product in $\mathbf{R}^n$ and $S_k^{[m-1]} = U_k^{[m]} - U_k^{[m-1]}$. We apply the two-point step size gradient method [9,28]

$$\begin{cases} U_k^{[m+1]} = U_k^{[m]} - \eta_k DJ_k\left(U_k^{[m]}\right), \quad m \geq 1, \quad P_k^{-1} U_k^{[m]} \in \mathbf{V}_k \\ \eta_k = \dfrac{DJ_k\left(U_k^{[m]}\right) S_k^{[m-1]}}{\langle S_k^{[m-1]}, S_k^{[m-1]} \rangle}, \\ U_k^{[0]} = P_k I_k \hat{u}_{k-1}, \qquad k \geq 1 \end{cases} \tag{13}$$

in order to find the solution of the unconstrained minimization problem

$$M_k : \underset{V_k \in \mathbf{V}_k}{\operatorname{argmin}} J_k(V_k), \qquad P_k^{-1} V_k \in \mathbf{V}_k.$$

The BB method has been successfully applied by Todorov [32,33] for solving finite element equations resulting from elliptic nonlocal problems.

Since $J_k(V_k)$ is a quadratic functional the iteration (13) becomes

$$\begin{cases} U_k^{[m+1]} = U_k^{[m]} - \eta_k \left(Q_k U_k^{[m]} - F_k\right), \quad m \geq 1, \quad P_k^{-1} U_k^{[m]} \in \mathbf{V}_k \\ \eta_k = \dfrac{\left(S_k^{[m-1]}\right)^T Q_k S_k^{[m-1]}}{\langle S_k^{[m-1]}, S_k^{[m-1]} \rangle}, \\ U_k^{[0]} = P_k I_k \hat{u}_{k-1}, \qquad k \geq 1 \end{cases} . \tag{14}$$

Thus, we obtain a multigrid iterative method for solving the problem (W_k) when the matrix D_k is positive definite.

Further, we suppose that the matrix D_k is indefinite or singular. There are various papers [3,4,18] that solve the singular linear system $AX = B$ based on the Hermitian and skew-Hermitian splitting. But the authors of the most of them require the matrix A to be positive semidefinite [7]. This requirement is too restrictive and cannot be satisfied in our case. Makinson and Shah [20] have proposed another splitting that avoids the requirement about positive semidefiniteness of the matrix A but they have imposed the condition the matrix A to have a workable split. Unfortunately, the authors have not presented a practical approach for constructing an arbitrary workable split of a singular matrix. A less restrictive method, which is easy to implementation has been developed by Srivastava et al. [30].

In this paper, we solve the linear system arising from the finite element discretization by an extrapolated iterative method proposed by Song and Wang [29]. Let X be a square matrix with real entries. We denote the spectral radius, the range space, the index and the spectrum of the matrix X by $\rho(X), R(X), \mathrm{Ind}(X)$ and $\sigma(X)$. The identity matrix is denoted by I and the Euclidean norm in $\mathbf{R}^n$ by $||\cdot||$.

We suppose that $\mathrm{Ind}(D_k) = 1$. Then the system (10) is very suitable for solving by the extrapolated method [29] since A_k is a symmetric and positive definite matrix. Since the problem (1-5) has at least one weak solution [15, Theorem 3.1] the vector $F_k \in R(D_k)$. We introduce the following iterative scheme

$$\begin{cases} U_k^{[m+1]} = Y_{\xi,k} U_k^{[m]} + \xi G_k, \\ Y_{\xi,k} = (1-\xi) I + \xi Y_k, \\ U_k^{[0]} = P_k I_k \hat{u}_{k-1}, \ \ k \geq 1 \end{cases} \tag{15}$$

in order to approximate the solution u_k of the problem (W_k). The matrices

$$Y_k = A_k^{-1}(B_k + C_k) \text{ and } G_k = A_k^{-1} F_k$$

are obtained from the splitting $D_k = A_k - (B_k + C_k)$. To start the iteration (15), we solve the problem (W_0) in the coarsest triangulation τ_0 by any method. We denote the approximate solution of (15) in the grid τ_k by $\hat{u}_k$.

The semiconvergence of the iterative method (15) is guaranteed by the next theorem.

Theorem 1 Let

$$\Lambda(\lambda) = \frac{2(1 - \mathrm{Re}\lambda)}{1 - 2\mathrm{Re}\lambda + |\lambda|^2}, \qquad \lambda \in \hat{\sigma}(Y), \qquad \hat{\sigma}(Y) = \sigma(Y) \setminus \{1\}$$

and

$$\underline{\gamma}_k = \min_{\lambda \in \hat{\sigma}(Y_k)} \Lambda(\lambda), \quad \overline{\gamma}_k = \max_{\lambda \in \hat{\sigma}(Y_k)} \Lambda(\lambda)$$

for an arbitrary square matrix Y. Additionally, we assume that $\mathrm{Ind}(D_k) = 1$. Then the iterative solution method (15) is semiconvergent when one of the conditions:

$$0 < \xi < \underline{\gamma}_k \text{ if } \mathrm{Re}\lambda < 1, \forall \lambda \in \hat{\sigma}(Y_k) \text{ or } \rho(Y_k) = 1;$$

$$\overline{\gamma}_k < \xi < 0 \text{ if } \mathrm{Re}\lambda > 1, \forall \lambda \in \hat{\sigma}(Y_k)$$

is true.

Proof. The proof is a direct consequence of [29, Theorem 2.2].■

Corollary 1 Under the conditions of Theorem 1, the convergence

$$\begin{cases} u_k^{[m]} \rightharpoonup u_k \\ m \to \infty \end{cases}$$

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holds and $u_k^{[\infty]}$ is independent of the initial guess $u_k^{[0]}$.
Proof. Theorem 1 assures the semiconvergence of the sequence $\left\{U_k^{[m]}\right\}$. Additionally, $U_k^{[\infty]}$ is independent of the initial guess. It remains to approach m to infinity in (15) with $u_k^{[m]} = P_k^{-1} U_k^{[m]}$ in order to obtain that $u_k^{[\infty]}$ is a solution of (W_k). ■

The essence of the method is presented in Algorithm 1. For the sake of simplicity this algorithm is constructed by assuming that the matrix D_k is singular.

Algorithm 1 *Multigrid algorithm for solving the telegraph equation*

1: % The initial guesses can be chosen in different ways.
2: In the coarsest grid, find a solution $\widehat{U}_0$ of the problem $\left(\widehat{W}_k\right)$ by any method and go to line 20:,
    or continue with the next line;
3: **compile** A_0, B_0, C_0, F_0;
4: **if** $\left(\mathrm{Re}\lambda < 1,\ \forall\lambda \in \hat{\sigma}(Y_0)\right)$ **or** ($\rho(Y_0) = 1$) **then**
5:   **choose** ξ so that $0 < \xi < \underline{\gamma}_0$
6: **end if**;
7: **if** $\left(\mathrm{Re}\lambda > 1,\ \forall\lambda \in \hat{\sigma}(Y_0)\right)$ **then**
8:   **choose** ξ so that $\overline{\gamma}_0 < \xi < 0$
9: **end if**;
10: $G_0 = A_0^{-1} F_0$;
11: $Y_0 = A_0^{-1}(B_0 + C_0)$;
12: $m = 0$;
13: **with** zero initial guess $U_0^{[0]}$
14: **repeat**
15:   $Y_{\xi,0} = (1-\xi)I + \xi Y_0$;
16:   $U_0^{[m+1]} = Y_{\xi,0} U_0^{[m]} + \xi G_0$;
17:   $m = m + 1$
18: **until** $\left\| U_0^{[m+1]} - U_0^{[m]} \right\| < \varepsilon$;
19: $\widehat{U}_0^{[m]} = U_0^{[m+1]}$;
20: **for** $k = 1$ **to** n
21:   $U_k^{[0]} = \widehat{U}_{k-1}$;
22:   **compile** A_k, B_k, C_k, F_k;
23:   $G_k = A_k^{-1} F_k$;
24:   $Y_k = A_k^{-1}(B_k + C_k)$;
25:   **if** $\left(\mathrm{Re}\lambda < 1, \forall\lambda \in \hat{\sigma}(Y_k)\right)$ **or** $\rho(Y_k) = 1$ **then**
26:     **choose** ξ so that $0 < \xi < \underline{\gamma}_k$;
27:   **end if**;
28:   **if** $\left(\mathrm{Re}\lambda > 1,\ \forall\lambda \in \hat{\sigma}(Y_k)\right)$
29:     **choose** ξ so that $\overline{\gamma}_k < \xi < 0$
30:   **end if**;
31:   $m = 0$;
32:   **repeat**
33:     $Y_{\xi,k} = (1-\xi)I + \xi Y_k$;
34:     $U_k^{[m+1]} = Y_{\xi,k} U_k^{[m]} + \xi G_k$;
35:     $m = m + 1$

36: **until** $\left\| U_k^{[m+1]} - U_k^{[m]} \right\| < \varepsilon$;
37: $U_k = U_k^{[m+1]}$
38: **end for**
39: **end.**

Remark 2 If $A_k > B_k$, $k \in \mathbf{N}$ the problem $(\widehat{W}_k)$ becomes stiffness dominated since $(V_k)^T C_k V_k = \underline{0}$ for all n_k-dimensional column vector V_k. Therefore, it can be solved by the fixed-point iteration method as well.

4. Numerical simulations

Let

$$e_k = \frac{\|P_k I^k u - P_k \hat{u}_k\|}{\sqrt{n_k}}$$

be the RMS error in the approximate solution $\hat{u}_k$ obtained by the triangulation τ_k. We use the approximate ARC [1,17]

$$\mu_k = \frac{1}{\ln 2} \ln \frac{e_{k-1}}{e_k}$$

in order to illustrate the convergence of the multigrid approximations.

In this section we approximately solve the linear telegraph equation

$$u_{tt} - u_{xx} + 2u_t + \tfrac{7}{4}\pi^2 u = f, \text{in } \Omega \tag{16}$$

provided with the periodic boundary conditions:

$$u(0,t) = u(1,t) = 0, t \in [0,1], \tag{17}$$

$$u(x,0) = u(x,1), x \in [0,1], \tag{18}$$

$$u_t(x,0) = u_t(x,1), x \in [0,1], \tag{19}$$

where Ω is the unit square and

$$f(z) = \pi \left(3 \sin(2\pi t) \sin \frac{3\pi x}{2} + 4x \cos(2\pi t) \cos \frac{3\pi x}{2} \right).$$

The weak form of the problem (16)-(19) looks as follows

$$(\nabla u, \nabla v) + \frac{7}{4}\pi^2 (u,v) = 2(u_t, v_t) + (u, v_t) - (u_t, v) + (f, v).$$

The problem (20) is solved by means of 10-node cubic triangular finite elements. We uniformly divide Ω into 18 right isosceles triangles. By unifying them we obtain the coarsest triangulation τ_0. The division process is continued by a refinement strategy [32] in order to obtain a sequence of hierarchical triangulations containing 18, 162, 1458, etc. finite elements. The matrices D_k and $\widehat{D}_k$ are positively defined and invertible, respectively. This is why we apply the preconditioned iterative method (14) in order to solve the problem (16)-(19). Following Raydan [28] and Todorov [32], we establish that the convergence of the two-point step size gradient method (14) depends on the condition number $\kappa(\widehat{D}_k)$. The smoothing of the error function $R_k u = I^k(u - \hat{u}_k)$ is illustrated in Figures 1-2. Table 1 represents the error in approximate solutions and the approximate ARC. The graphs of the approximate and the exact solutions are presented in Figure 3. The solution $\hat{u}_0$ is obtained in triangulation τ_0 by only 18 cubic finite elements. Despite this both graphics in Figure 3 are indistinguishable.

Table 4. The error in the approximate solutions and the approximate ARC.

k	$\mathrm{Card}(\tau_k)$	e_k	μ_k
0	18	0.00347540006294	
1	162	0.00017974078023	4.27319
2	1458	0.00001071379797	4.06838

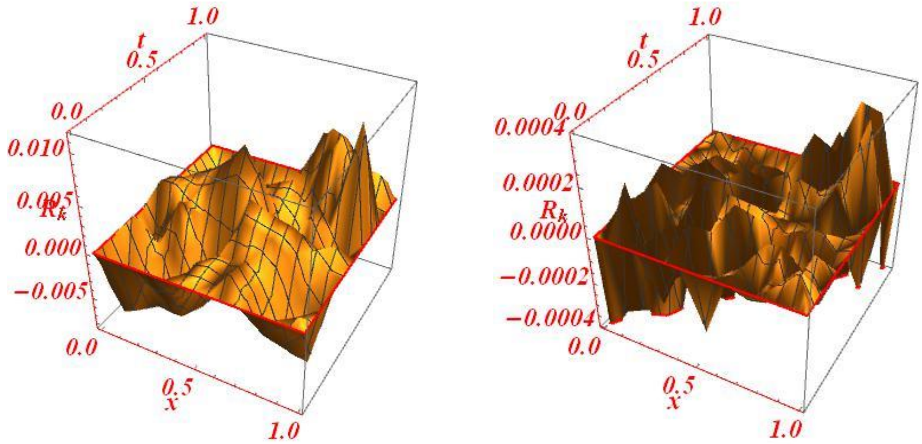

Figure 1. The graph of the error function R_k obtained by 18 finite elements *left* and by 162 elements *right*.

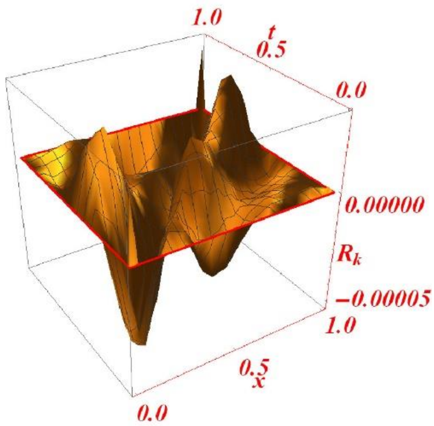

Figure 2. The graph of the error function R_k obtained by 1458 elements.

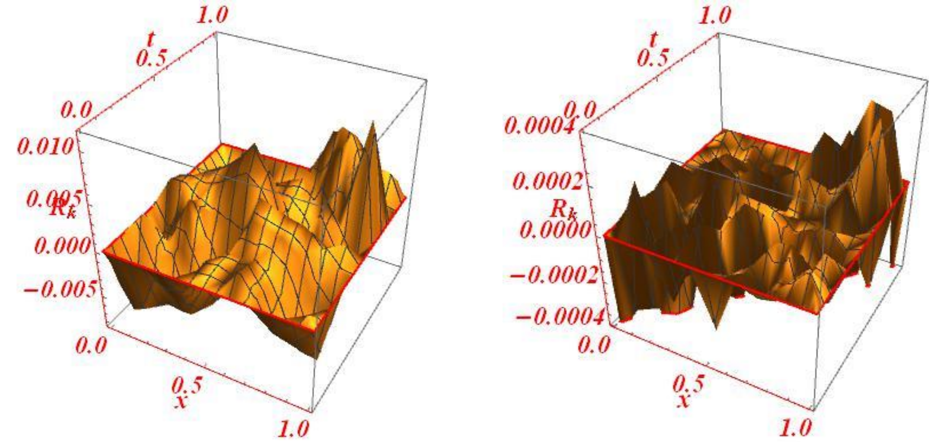

Figure 3. The graphs of the approximate solution $\hat{u}_0$ *left* and the exact solution u *right*.

5. Conclusion

This paper deals with a space-time finite element method for solving the linear telegraph equation. The proposed method should not only be restricted to solving a telegraph equation. It can be successfully applied to solve other hyperbolic problems. The multigrid method for solving the problem (W) is based on the simplicial elements. But it can be successfully used with quadrilateral elements as well. Since the strong problem could have more than one solution, we consider different iterative methods for solving the system of linear finite element equations. The case when the original problem has more than one solution is considered more thoroughly by describing the necessary iterative method in pseudocode.

The numerical simulations of the RMS error indicate a quartic ARC for the cubic trial functions. Due to ill-conditioned linear systems of finite element equations, preconditioned techniques are used. Todorov [32,33] has demonstrated slight dependence on the convergence of the BB method on the initial guesses solving elliptic nonlocal problems. Unfortunately, the method (14) is sensible with respect to the initial guesses. The latter means that the initial guesses should be chosen close to the weak solution. That is why we need an approximate solution in the coarsest triangulation in order to start the multigrid algorithm.

References

[1] Andreev A B, Todorov T D 2006 Isoparametric Finite Element Approximation for a Boundary Flux Problem, *The International Journal of Numerical Methods for Heat & Fluid Flow* **16**, 1, 46-66.

[2] Arslan D 2020 The Numerical Study of a Hybrid Method for Solving Telegraph Equation, Applied Mathematics and Nonlinear Sciences **5**,1, 293-302.

[3] Bai Z-Z, Golub G H, Pan J-Y 2004 Preconditioned Hermitian and skew-Hermitian splitting methods for non-Hermitian positive semidefinite linear systems, *Numer. Math.* **98** 1-32.

[4] Bai Z-Z 2010 On semi-convergence of Hermitian and skew-Hermitian splitting methods for singular linear systems, *Computing* **89** 171-197.

[5] Berwal N, Panchal D, Parihar C L 2013 Haar wavelet method for numerical solution of telegraph equations, *Italian Journal of Pure and Applied Mathematics* **30** 317-328.

[6] R. G. Campos 2014 Numerical simulation of reaction-diffusion telegraph systems in unbounded domains, *Numerical Methods for Partial Differential Equations* **31**, 1, 326-335.

[7] Cao Z-H 2002 On the convergence of iterative methods for solving singular linear systems, Journal of Computational and Applied Mathematics **145** 1-9.

[8] Chen C, Liu W, Zhao X 2014 A Two-Grid Finite Element Method for a Second-Order Nonlinear Hyperbolic Equation, *Abstract and Applied Analysis*, **2014**, Article ID 803615, 1-6.

[9] Cruz W La 2009 Extension of the Barzilai-Borwein Method for Quadratic Forms in Finite Euclidean Spaces, *Numerical Functional Analysis and Optimization* **30,** 3-4 306-321.

[10] Dehghan M, Ghesmati A 2010 Combination of meshless local weak and strong (MLWS) forms to solve the two dimensional hyperbolic telegraph equation, *Engineering Analysis with Boundary Elements* **34**, 51-59.

[11] Ghoussoub N, Moradifam A 2011 A note on simultaneous preconditioning and symmetrization of non-symmetric linear systems, *Numerical Linear Algebra with Applications* **18**, 3, 343-349.

[12] Grossinho M R, Nkashama M N 1998 Periodic solutions of parabolic and telegraph equations with asymmetric nonlinearities, *Nonlinear Analysis: Theory, Methods & Applications* **33**, 2, 187-210.

[13] Hafez R M 2018 Numerical solution of linear and nonlinear hyperbolic telegraph type equations with variable coefficients using shifted Jacobi collocation method, *Comp. Appl. Math.* **37**, 5253-5273.

[14] Kharibegashvili S S, Dzhokhadze O M 2015 Time-Periodic Problem for a Weakly Nonlinear Telegraph Equation with Directional Derivative in the Boundary Condition, *Differential Equations* **51**, 10, 1369-1386.

[15] Kim W S 1988 Boundary value problem for nonlinear telegraph equations with superlinear growth, *Nonlinear Analysis: Theory, Methods & Applications*, **12**, 1371-1376.

[16] Larsson S, Thomée V 2003 The Finite Element Method for Hyperbolic Equations. In: *Partial Differential Equations with Numerical Methods. Texts in Applied Mathematics*, **45** Springer, Berlin, Heidelberg.

[17] Li H, Mazzucato A, Nistor V 2010 Analysis of the finite element method for transmission/mixed boundary value problems on general polygonal domains, *Electronic Transactions on Numerical Analysis*, **37**, 41-69.

[18] Li W, Liu Y-P, Peng X-F 2012 The generalized HSS method for solving singular linear systems, *Journal of Computational and Applied Mathematics* **236,** 2338-2353.

[19] Lin C, Gu M, Young D 2010 The time-marching method of fundamental solutions for multi-dimensional telegraph equations, *CMC-Computers, Materials & Continua* **18**, 1, 43-68.

[20] Makinson G J, Shah A A 1986 An iterative solution method for solving sparse nonsymmetric linear systems, *Journal of Computational and Applied Mathematics* **15**, 339-352.

[21] Mawhin J 2005 Maximum Principle for Bounded Solutions of the Telegraph Equation: The Case of high dimensions, *Progress in Nonlinear Differential Equations and Their Applications*, Birkhäuser Verlag, Basel, Switzerland **63**, 343-351.

[22] Mawhin J, Ortega R, Robles-Pérez A M 2000 A Maximum Principle for Bounded Solutions of the Telegraph Equations and Applications to Nonlinear Forcings, *Journal of Mathematical Analysis and Applications* **251**, 695-709.

[23] Mawhin J, Ortega R, Robles-Pérez A M 2005 Maximum principles for bounded solutions of the telegraph equation in space dimensions two and three and applications, *J. Differential Equations* **208**, 42-63.

[24] Nagaveni K 2020 Haar Wavelet Collocation Method for Solving the Telegraph Equation with Variable Coefficients, *International Journal of Applied Engineering Research* **15**, 3, 235-243.

[25] Nazir T, Abbas M, Yaseen M 2017 Numerical solution of second-order hyperbolic telegraph equation via new cubic trigonometric B-splines approach, *Cogent Mathematics* **4**, 1, 1382061.

[26] Ortega R, Robles-Pérez A M 1998 A Maximum Principle for Periodic Solutions of the Telegraph Equation, *Journal of Mathematical Analysis and Applications* **221**, 625-651.

[27] Pandit S, Kumar M, Tiwari S 2015 Numerical simulation of second-order hyperbolic telegraph type equations with variable coefficients, Computer Physics Communications **187**, 83-90.

[28] Raydan M 1997 The Barzilai and Borwein gradient method for the large scale unconstrained minimization problem, *SIAM Journal on Optimization* **7**, 1, 26-33.

[29] Song Y, Wang L 2003 On the semiconvergence of extrapolated iterative methods for singular linear systems, *Applied Numerical Mathematics* **44** 401-413.

[30] Srivastava S, Gupta D K, Singh A 2016 An iterative method for solving singular linear systems with index one, *Afr. Mat.* **27**, 815-824.

[31] Shivanian E, Abbasbandy S, Khodayari 2018 A Numerical simulation of 1D linear telegraph equation with variable coefficients using meshless local radial point interpolation (MLRPI), *Int. J. Industrial Mathematics* **10**, 2, 1-14.

[32] Todorov T D 2018 Dirichlet Problem for a Nonlocal p-Laplacian Elliptic Equation, *Computers & Mathematics with Applications* **76**, 6, 1261-1274.

[33] Todorov T D 2015 Nonlocal problem for a general second-order elliptic operator, *Computers & Mathematics with Applications* **69**, 5, 411-422.

[34] Zhang B, Yu W, Mascagni M 2019 Revisiting Kac's method: A Monte Carlo algorithm for solving the Telegrapher's equations, *Mathematics and Computers in Simulation* **156**,178-193.

[35] Zhao Z, Li H, Liu Y 2020 Analysis of a continuous Galerkin method with mesh modification for two-dimensional telegraph equation, *Computers & Mathematics with Applications* **79**, 3, 588-602.